GAMBLING PROBLEMS WITH A LIMIT INFIMUM PAYOFF

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Abstract

In the gambling theory of Dubins and Savage, the payoff from a sequence of states is the limit supremum of their utilities. Here the limit infimum is used instead. For example, if the utility function is the indicator of a set G, then the Dubins and Savage payoff is 1 when infinitely many states are in G, while the payoff here is 1 when all but finitely many states are in G. There are some resulting changes in the theory, but it remains true that optimal stationary plans exist for finite problems.

Key words: gambling, dynamic programming, optimal stationary plan.

1. Introduction

Let (F,Γ,u) be a <u>classical gambler's problem</u> as defined by Dubins and Savage [3]. That is, F is a nonempty set and, for each $f \in F$, $\Gamma(f)$ is a nonempty set of probability measures (called <u>gambles</u>) defined on subsets of F. The <u>utility function</u> u is a bounded function from F into the real numbers. As is explained in [3], a gambler with initial fortune f selects a strategy σ (available) at f. The strategy determines the distribution (also denoted by σ) of the stochastic process $f_1, f_2, \ldots$ of the gambler's future fortunes. The payoff to the gambler is defined to be

$$(1.1) \qquad u(\sigma) = \overline{\lim_t} \int u(f_t)\, d\sigma$$

where the lim sup is taken over the directed set of stop rules and the optimal return function is, for each $f \in F$,

$$(1.2) \qquad V(f) = \sup\{u(\sigma)\colon \sigma \text{ at } f\}.$$

As explained by Dubins and Savage, they define $u(\sigma)$ as above because they "do not intend to appraise σ in terms of fortunes visited early, nor ... to penalize σ for visits to fortunes of low utility, however frequent, if these serve as stepping stones to fortunes of high utility."

The present paper is a study of the payoff defined by

$$(1.3) \qquad \underline{u}(\sigma) = \underline{\lim_t} \int u(f_t)\, d\sigma.$$

The corresponding optimal return function is

$$(1.4) \qquad W(f) = \sup\{\underline{u}(\sigma)\colon \sigma \text{ at } f\}.$$

The payoff $\underline{u}(\sigma)$ also does not appraise σ in terms of fortunes visited early. However, it does penalize σ for visiting fortunes of low utility too frequently.

There are alternative formulas for $u(\sigma)$ and $\underline{u}(\sigma)$ which may help to clarify their differences. Let $H = F^N$ be the set of all sequences of elements of F, and, for $h = (f_1, f_2, \ldots) \in H$, define

$$u^*(h) = \overline{\lim_n}\, u(f_n), \quad u_*(h) = \underline{\lim_n}\, u(f_n). \tag{1.5}$$

Then, by [2, Theorem 1] or [11, Theorem 3.2],

$$\begin{aligned} u(\sigma) &= \int u^*\, d\sigma \\ \underline{u}(\sigma) &= \int u_*\, d\sigma. \end{aligned} \tag{1.6}$$

In the special case when u is the indicator of a set $G \subset F$, these formulas become

$$\begin{aligned} u(\sigma) &= \sigma[f_n \in G \text{ i.o.}] \\ \underline{u}(\sigma) &= \sigma[f_n \in G \text{ eventually}], \end{aligned} \tag{1.7}$$

where "i.o." is short for "infinitely often" and "eventually" means "all but a finite number of n".

One of the most interesting theorems [3, Theorem 3.9.1] in the book by Dubins and Savage states that if F is finite and if $\Gamma(f)$ is finite for every f, then there is an optimal stationary plan for the lim sup payoff. The major result of this note (Theorem 3.1 below) is that the same theorem holds for the lim inf payoff. There are, however, payoff functions for which the result fails. A simple example is in section 5.

If Γ is leavable in the sense that the point mass $\delta(f)$ is in $\Gamma(f)$ for all f, then $W = V$ (Proposition 4.1 below). Also, for any leavable problem with a bounded u, if ε-optimal stationary plans are available for every $\varepsilon > 0$ for the lim sup problem, they are also available for the lim inf problem (Theorem 4.2). A number of results already available for the lim sup problem can thus be carried over to the lim inf problem (cf. [5]).

Unless the contrary is stated, notation and terminology are as in Dubins and Savage [3].

2. Preliminary lemmas.

This section contains a number of lemmas, including a characterization of optimal strategies, which will be used in the construction of stationary plans in section 3. These results hold in the finitely additive setting of [3] and [7], and in the measurable, countably additive setting of [4] and [12]. Proofs will be presented for the finitely additive theory, but, with the addition of some measurability assumptions like those of [4] and [12], are valid for the countably additive theory as well. The major interest is in the finite case where the two theories coincide.

Let $\Sigma(f)$ denote the collection of strategies σ available at f in Γ and let $\Sigma = \cup\{\Sigma(f)\colon f \in F\}$. For every σ, define

$$W(\sigma) = \int W^* \, d\sigma = \overline{\lim_t} \int W(f_t) \, d\sigma.$$

Although u_* is the payoff function studied in most of this paper, the first four lemmas are true for any payoff function $g\colon H \to R$ which is bounded, measurable in the sense of [8], and shift-invariant. The proof of Lemma 1 below is essentially the same as that of Theorems 6(b) and 7 in [12], the difference being that one must use finitely additive martingale convergence theorems from [7] and [8].

<u>Lemma 1</u>. For every $f \in F$ and $\sigma \in \Sigma(f)$, the following hold:

(a) $W(f), W(f_1), \ldots$ is a bounded supermartingale under σ amd, hence, $W(f_n) \to W^* = W_*$ σ-almost surely.

(b) $W(f) \geq W(\sigma) \geq \underline{u}(\sigma)$.

(c) $\sigma[u_* > W^*] = 0$.

A $\sigma \in \Sigma(f)$ is called <u>thrifty</u> if equality holds in the first inequality of (b) and <u>equalizing</u> if equality holds in the second inequality of (b). Thus σ is optimal if and only if it is both thrifty and equalizing.

For $\varepsilon \geq 0$, let Γ_ε be the subhouse of Γ containing the ε-conserving gambles; that is,

$$\Gamma_\varepsilon(f) = \{\gamma \in \Gamma(f): \gamma W \geq W(f) - \varepsilon\}$$

for every $f \in F$, and let W_ε be the corresponding optimal return function. The next lemma states that, if $\varepsilon > 0$, then a gambler who must use only ε-conserving gambles is not harmed.

<u>Lemma 2</u>. For every $\varepsilon > 0$, $W_\varepsilon = W$.

<u>Proof</u>: Let $0 < \delta < \varepsilon$ and let $f \in F$. Choose $\sigma \in \Sigma(f)$ such that

$$\underline{u}(\sigma) \geq W(f) - \delta^2.$$

If t is a stop rule, then, by Lemma 1 and [3, formula 3.6.1],

$$\int W(f_t)\, d\sigma \geq W(\sigma) \geq W(f) - \delta^2$$

and it follows from [10, Theorem 3] that σ uses only gambles available in Γ_ε up to time t with probability at least $1 - \delta$.

Let σ' be a strategy available in Γ_ε at f such that σ' uses the same gambles as σ whenever those gambles are available in Γ_ε. Then, for every t, σ' agrees with σ up to time t with probability at least $1 - \delta$. One can prove, by induction on the structure of f_t, that

$$\left|\int u(f_t)\, d\sigma - \int u(f_t)\, d\sigma'\right| \leq 2\delta M$$

where $M = \sup\{|u(f)|: f \in F\}$. (With only slightly more effort, one shows $\left|\int g\, d\sigma - \int g'\, d\sigma\right| \leq 2\delta \sup|g|$ for all measurable g.) Hence, $\underline{u}(\sigma') \geq \underline{u}(\sigma) - 2\delta M$.□

A problem (F,Γ,u) is <u>finite</u> if F is finite and, for every f, $\Gamma(f)$ is finite. For such problems, there is no loss to a gambler restricted to 0-conserving gambles.

<u>Lemma 3</u>. For finite problems, $W_0 = W$.

<u>Proof</u>: For some $\varepsilon > 0$, $\Gamma_0 = \Gamma_\varepsilon$ and, hence, $W_0 = W_\varepsilon$. But $W_\varepsilon = W$ by by Lemma 2.□

The next lemma should suggest why it is desirable to restrict attention to 0-conserving gambles. A strategy which always uses them is thrifty.

<u>Lemma 4</u>. If $\Gamma_0 = \Gamma$, then, for every $\sigma \in \Sigma$, $W(f) = W(\sigma)$.

<u>Proof</u>: Same as in [3, Theorem 3.6.1] or [12, Theorem 10].

The next lemma gives a characterization of equalizing strategies analogous to that of Dubins and Savage for the lim sup payoff [3, Theorem 3.7.2]. For $\varepsilon \geq 0$, let v^ε be the indicator function of the set $\{f: u(f) \geq W(f) - \varepsilon\}$.

Lemma 5. For $f \in F$ and $\sigma \in \Sigma(f)$, $W(\sigma) = \underline{u}(\sigma)$ if and only if $\underline{v}^\varepsilon(\sigma) = 1$ for all $\varepsilon > 0$.

Proof: Suppose $\underline{v}^\varepsilon(\sigma) = 1$ for $\varepsilon > 0$. By (1.7),

$$u(f_n) \geq W(f_n) - \varepsilon$$

for all but finitely many n with σ-probability 1. Hence,

$$u_* \geq W^* - \varepsilon \quad \sigma\text{-almost surely}$$

and, consequently,

$$\underline{u}(\sigma) = \int u_* \, d\sigma \geq \int W^* \, d\sigma - \varepsilon = W(\sigma) - \varepsilon.$$

It follows from Lemma 1 that $\underline{u}(\sigma) = W(\sigma)$.

Now assume $\underline{u}(\sigma) = W(\sigma)$. That is,

$$\int (W^* - u_*) \, d\sigma = 0.$$

It now follows from Lemma 1(c) that

$$\sigma[W^* - u_* \leq \varepsilon] = 1 \tag{2.1}$$

for all $\varepsilon > 0$.

But

$$W^*(h) - u_*(h) = \overline{\lim_n}(W(f_n) - u(f_n))$$

and, hence, $W^*(h) - u_*(h) \leq \varepsilon$ implies that $u(f_n) \geq W(f_n) - 2\varepsilon$ for all but finitely many n. Thus (2.1) implies that $\underline{v}^{2\varepsilon}(\sigma) = 1$.□

Lemma 6. If F is finite, $f \in F$, and $\sigma \in \Sigma(f)$, then σ is equalizing iff

$$\sigma\{h: u(f_n) \geq W(f_n) \text{ eventually}\} = 1.$$

Proof: Let

$$S = \{f: u(f) < W(f)\}$$

and

$$\varepsilon = \min\{W(f) - u(f): f \in S\}.$$

Then $\{f: u(f) \geq W(f)\} = \{f: u(f) \geq W(f) - \varepsilon/2\}$. Now use Lemma 5.□

For a classical gambler's problem, equalizing strategies always exist [3, Theorem 3.8.4]. This is not true for lim inf problems as the following example illustrates.

Example: Let $F = \{1,2,\ldots\}$; $\Gamma(n) = \{\delta(n),\delta(n+1)\}$; $u(n) = 1 - n^{-1}$ for n odd, $u(n) = 0$ for n even. Then $W \equiv 1$ and there are no equalizing strategies available.

Of course, "ε-equalizing strategies" must exist and the next lemma states that slightly more is true.

Let W^ε be the optimal return function associated with the lim inf payoff for the problem (F,Γ,v^ε).

Lemma 7. If $\varepsilon > 0$, then $W^{\varepsilon} \equiv 1$.

Proof: Let $\delta > 0$ and $f \in F$. Choose $\sigma \in \Sigma(f)$ such that $\underline{u}(\sigma) \geq W(f) - \delta^2$. By Lemma 1(b), $\underline{u}(\sigma) \geq W(\sigma) - \delta^2$ from which it follows, by an argument like that used in the second half of the proof of Lemma 5, that

$$\int (W^* - u_*)\, d\sigma \leq \delta^2$$

and, hence, for $\delta < \varepsilon/2$,

$$\sigma[W^* - u_* < \varepsilon/2] \geq 1 - \delta.$$

Thus

$$\underline{v}^{\varepsilon}(\sigma) \geq 1 - \delta. \square$$

Lemma 8. If F is finite, then $W^0 \equiv 1$.

Proof: For sufficiently small $\varepsilon > 0$, $v^0 = v^{\varepsilon}$. $\square$

The final lemma of this section, which was suggested by a remark of Manfred Schäl, is a $0 - 1$ law for gambling houses. It is most easily stated using the following notation of Hill [6]: For S a Borel subset of H in the sense of [7] and $f \in F$, define

$$P_f(S) = \sup\{\sigma(S): \sigma \in \Sigma(f)\},$$

and let

$$P(S) = \sup\{P_f(S): f \in F\}.$$

Lemma 9. If S is shift-invariant, then either $P(S) = 0$ or $P(S) = 1$.

Proof: Suppose $P(S) > 0$. So $\exists \sigma \in \Sigma$ such that $\sigma(S) > 0$. By the Lévy

$0 - 1$ law [8] and the shift-invariance of S,

$$\sigma[p_n(h)](Sp_n(h)) = \sigma[p_n(h)](S) \to 1$$

for $h \in S$ σ-almost surely.□

3. Optimal stationary plans for finite problems.

A Γ-*selector* is a function γ with domain F such that $\gamma(f) \in \Gamma(f)$ for all f. Such a selector γ determines a *stationary plan* or *family of strategies* γ^∞; for each f, $\gamma^\infty(f)$ is that strategy in $\Sigma(f)$ which uses $\gamma(f')$ whenever the current position is f'. The plan γ^∞ is *optimal for the lim inf payoff* if $\underline{u}(\gamma^\infty(f)) = W(f)$ for all f.

Theorem 1. For every finite problem (F,Γ,u), there is a stationary plan γ^∞ which is optimal for the lim inf payoff.

The rest of this section is devoted to the proof of Theorem 1. So assume, for the rest of this section, that the problem is finite.

By Lemma 2.3, it can be assumed that all gambles available are 0-conserving and so, by Lemma 2.4, that all strategies are thrifty. It remains to show that there is a selector γ such that $\gamma^\infty(f)$ is equalizing for all f.

Define

$$G = \{f: u(f) \geq W(f)\}$$

and

$$E = \{h: f_n \in G \text{ eventually}\}. \tag{3.1}$$

Then, by Lemma 2.6, it suffices to find γ such that, for all f,

$$\gamma^{\infty}(f)(E) = 1. \tag{3.2}$$

Recall that, in the notation of section 2,

$$W^0(f) = \sup\{\sigma(E): \sigma \in \Sigma(f)\},$$

and

$$P_f(G^N) = \sup\{\sigma(G^N): \sigma \in \Sigma(f)\}$$

where

$$G^N = \{h: f_n \in G \text{ for all } n\}.$$

Lemma 1. There exists an $f \in G$ such that $P_f(G^N) = 1$.

Proof: Let $\varepsilon > 0$. Because F is finite, it is enough to find an f such that $P_f(G^N) > 1 - \varepsilon$.

By Lemma 2.8, $\exists \sigma \in \Sigma$ such that $\sigma(E) > 1 - \varepsilon$. Also

$$E = \bigcup_1^{\infty} C_n$$

where

$$C_n = \{h: f_k \in G \text{ for all } k \geq n\}.$$

Since $C_n \uparrow E$ and σ is countably additive, $\exists m \ni \sigma(C_m) > 1 - \varepsilon$.

Finally,

$$\sigma(C_m) = \int_G \sigma[p_m](G^N)\, d\sigma$$

so that, for some h, $f_m \in G$ and $\sigma[p_m(h)](G^N) > 1 - \varepsilon$. Take $f = f_m$.□

Define

$$G_0 = \{f \in G: P_f(G^N) = 1\}.$$

Lemma 2. For every $f \in G_0$, $\exists\gamma \in \Gamma(f)$ such that $\gamma(G_0) = 1$.

Proof: Let $f \in G_0$ and suppose, to get a contradiction, that $\gamma(G_0) \leq 1 - \varepsilon$ for some $\varepsilon > 0$ and all $\gamma \in \Gamma(f)$.

Consider first the case when $G = G_0$. Then, for $\sigma \in \Sigma(f)$,

$$\sigma(G^N) = \sigma(G_0^N) \leq \sigma_0(G_0) \leq 1 - \varepsilon,$$

which contradicts the assumption that $f \in G_0$.

Next consider the case when $G - G_0$ is not empty. Let

$$\alpha = \min\{1 - P_{f'}(G^N): f' \in G - G_0\}.$$

Then $\alpha > 0$ and, for all $f' \in G - G_0$ and $\sigma' \in \Sigma(f')$, $\sigma'(G^N) \leq 1 - \alpha$.

Consider next any $\sigma \in \Sigma(f)$ and calculate as follows:

$$\begin{aligned}\sigma(G^N) &= \int_{G_0} \sigma[f_1](G^N)\, d\sigma_0(f_1) + \int_{G-G_0} \sigma[f_1](G^N)\, d\sigma_0(f_1) \\ &\leq \sigma_0(G_0) + (1-\alpha)\sigma_0(G-G_0) \\ &= (1-\alpha)\sigma_0(G) + \alpha\sigma_0(G_0) \\ &\leq (1-\alpha) + \alpha(1-\varepsilon) \\ &= 1 - \alpha\varepsilon.\end{aligned}$$

Hence, $P_f(G^N) \leq 1 - \alpha\varepsilon$, a contradiction.□

It should be clear that Lemma 2 provides good gambles on G_0. The next lemma suggests that, at an f not in G_0, the problem reduces to one of reaching G_0. To state the lemma, recall (3.1) and define

$$E_0 = \{h: f_n \in G_0 \text{ eventually}\}.$$

Lemma 3. For every $\sigma \in \Sigma$, $\sigma(E) = \sigma(E_0)$.

Proof: Let $S = E \cap \{h: f_n \in G - G_0 \text{ i.o.}\}$. Then

$$E - E_0 \subset S$$

and it suffices to prove that $\sigma(S) = 0$ for all $\sigma \in \Sigma$. This follows from Lemma 2.9 and the following fact:

(3.3) $\sigma(S) \leq 1 - \alpha$, for all f and all $\sigma \in \Sigma(f)$,

where $\alpha = \min\{1 - P_f(G^N): f \in G - G_0\}$. (Assume $G - G_0$ is not empty since the result is obvious otherwise.)

To prove (3.3), let $\varepsilon > 0$ and fix $\sigma \in \Sigma$. Define

$$\beta(h) = \sup\{n: f_n \notin G\}.$$

Notice that $S \subset [\beta < \infty]$. So, by countable additivity of σ, there is a positive integer M such that

(3.4) $\sigma(S \cap [\beta \leq M]) \geq \sigma(S) - \varepsilon$.

Let

$$\tau(h) = \inf\{n: n \geq M, \ f_n \in G - G_0\}.$$

(The infimum of the empty set is taken to be ∞.) Then $S \subset [\tau < \infty]$. So there is a positive integer $N > M$ such that

$$(3.5) \qquad \sigma(S \cap [\beta \leq M] \cap [\tau \leq N]) \geq \sigma(S \cap [\beta \leq M]) - \varepsilon.$$

Let $A = S \cap [\beta \leq M] \cap [\tau \leq N]$ and calculate as follows:

$$(3.6) \qquad \sigma(A) = \int_{\tau \leq N} \sigma[p_\tau](Ap_\tau)\, d\sigma$$

$$\leq \int_{\tau \leq N} \sigma[p_\tau](G^N)\, d\sigma$$

$$\leq 1 - \alpha.$$

Here the equality is by [3, formula 3.7.1]; the first inequality holds because $\tau \geq \beta$ on A and, hence, $Ap_\tau \subset G^N$; the final inequality holds by definition of α together with the fact that $f_\tau(h) \in G - G_0$ for $h \in A$.

Inequality (3.3) follows from (3.4), (3.5), and (3.6).□

Consider now the classical gambler's problem (F,Γ,u_0) with lim sup payoff where u_0 is the indicator of G_0. Let V_0 be the corresponding optimal return function.

<u>Lemma 4</u>. For all $f \in F$, $V_0(f) = 1$.

<u>Proof</u>: Let $\sigma > 0$ and $f \in F$. By Lemma 2.8, there is $\sigma \in \Gamma(f)$ such that $\sigma(E) \geq 1 - \varepsilon$. Then, by definition of V_0, (1.7), and Lemma 3,

$$V_0(f) \geq u_0(\sigma) = \sigma[f_n \in G_0 \ \text{ i.o.}] \geq \sigma(E_0) = \sigma(E).□$$

The way is now prepared to complete the proof of (3.2) and with it the proof of Theorem 1.

By Theorem 3.9.1 of [3], there is a Γ-selector α such that

$$u_0(\alpha^{\infty}(f)) = V_0(f)$$

for all f. So, by Lemma 4 and (1.7),

$$\alpha^{\infty}(f)[f_n \in G_0 \quad \text{i.o.}] = 1$$

and, hence,

$$\alpha^{\infty}(f)[\tau < \infty] = 1 \tag{3.7}$$

where

$$\tau(h) = \inf\{n: f_n \in G_0\}.$$

By Lemma 2, there is a Γ-selector β such that

$$\beta(f)(G_0) = 1$$

for all $f \in G_0$. Hence,

$$\beta^{\infty}(f)(G_0^N) = 1 \tag{3.8}$$

for all $f \in G_0$.

Define the Γ-selector γ to equal β on G_0 and to equal γ on $F - G_0$. By (3.7) the stationary plan $\gamma^{\infty}(f)$ reaches G_0 with probability 1 from any $f \in F - G_0$ and, by (3.8), stays in G_0 forever with probability 1 after reaching it. Hence,

$$\gamma^{\infty}(f)(E_0) = \gamma^{\infty}(E) = 1$$

for all f.

The proof of Theorem 1 is now complete.

<u>Remark</u>: The method of proof for Theorem 1 obviously owes a great deal to that of Dubins and Savage [3, Theorem 3.9.1] and also has similarities to the proofs of Theorem 2 in [14] and Theorem 3 of [4].

4. <u>Stationary plans for leavable problems</u>.

Let $\mathcal{P}$ be a collection of plans for a gambling problem (F,Γ,u). Then $\mathcal{P}$ is <u>adequate for the lim sup (lim inf) payoff</u> if, for each $\varepsilon > 0$ and $f \in F$, $\exists\bar{\sigma} \in \mathcal{P}$ such that

(4.1) $\quad u(\bar{\sigma}(f)) \geq V(f) - \varepsilon$

(4.2) $\quad (\underline{u}(\bar{\sigma}(f)) \geq W(f) - \varepsilon)$.

If, for each $\varepsilon > 0$, $\exists\bar{\sigma} \in \mathcal{P}$ such that (4.1) (respectively (4.2)) holds for all f, then $\mathcal{P}$ is <u>uniformly adequate for the lim sup (lim inf) payoff</u>. For example, in finite problems the <u>collection</u> $\mathcal{S}$ of <u>stationary plans</u> is uniformly adequate in both senses by [3, Theorem 3.9.1] and Theorem 3.1.

<u>Theorem 2</u>. Let (F,Γ,u) be a classical gambler's problem for which Γ is leavable and u is bounded. If $\mathcal{S}$ is adequate (uniformly adequate) for the lim sup payoff, then $\mathcal{S}$ is adequate (uniformly adequate) for the lim inf payoff.

Theorem 2 will be an immediate consequence of Propositions 1 and 2 below.

Proposition 1. If Γ is leavable, then $W = V$.

Proof: Obviously, $W \leq V$. Also, W is excessive by Lemma 2.1 and $W \geq u$. Hence, $W \geq V$ by [3, Theorem 2.12.1] and [3, Corollary 3.3.2].□

Until the end of the proof of the next proposition, assume Γ is leavable and u is bounded in absolute value by the finite constant M. Let σ be in $\Sigma(f)$ and define, for $\varepsilon > 0$,

$$\tau(h) = \tau_\varepsilon(h) = \inf\{n: u(f_n) \geq V(f_n) - \varepsilon\}. \tag{4.3}$$

Define σ' to be that strategy which agrees with σ up to time τ and then stagnates. That is, for $h = (f_1, f_2, \ldots)$,

$$\begin{aligned} \sigma'_n(f_1,\ldots,f_n) &= \sigma_n(f_1,\ldots,f_n) && \text{if } \tau(h) > n, \\ &= \delta(f_n) && \text{if } \tau(h) \leq n. \end{aligned}$$

Proposition 2. If $u(\sigma) \geq V(f) - \varepsilon^2$, then $\underline{u}(\sigma') \geq V(f) - (\varepsilon^2 + 3\varepsilon M + \varepsilon)$.

The proof of Proposition 2 will be given in three lemmas.

Lemma 1. $\sigma[\tau < \infty] \geq 1 - \varepsilon$.

Proof: Choose a stop rule t such that

$$\int u(f_t)\, d\sigma \geq V(f) - \varepsilon^2, \tag{4.4}$$

as is possible under the hypothesis of Proposition 2. Let $A = [u(f_t) \geq V(f_t) - \varepsilon]$. Then

$$(4.5) \qquad \int u(f_t)\, d\sigma = \int_A u(f_t)\, d\sigma + \int_{A^c} u(f_t)\, d\sigma$$

$$\leq \int_A V(f_t)\, d\sigma + \int_{A^c} (V(f_t) - \varepsilon)\, d\sigma$$

$$\leq V(f) - \varepsilon\sigma(A^c),$$

where the equality is obvious, the first inequality because $u \leq V$ for leavable Γ, and the second because V is excessive [3, Theorem 3.4.1.].

By (4.4) and (4.5),

$$\sigma(A^c) \leq \varepsilon.$$

Hence,

$$\sigma[\tau < \infty] \geq \sigma(A) \geq 1 - \varepsilon. \square$$

<u>Lemma 2</u>. $u(\sigma') \geq V(f) - (\varepsilon^2 + \varepsilon M + \varepsilon)$.

<u>Proof</u>: This is a consequence of [9, Lemma 3.2].□

<u>Lemma 3</u>. $\underline{u}(\sigma') \geq u(\sigma') - 2\varepsilon M$.

<u>Proof</u>: For every stop rule t and τ as in (5.3),

$$\int u(f_t)\, d\sigma' = \int_{\tau \leq t} u(f_\tau)\, d\sigma + \int_{\tau > t} u(f_t)\, d\sigma$$

because σ and σ' agree up to time $\tau \wedge t$ and [9, Lemma 3.3] applies.

Hence,

$$\underline{u}(\sigma') = \underline{\lim}_t \int u(f_t)\, d\sigma'$$

$$\geq \int_{\tau < \infty} u(f_\tau)\, d\sigma - \varepsilon M,$$

and

$$u(\sigma') = \overline{\lim_{t}} \int u(f_t)\, d\sigma'$$

$$\leq \int_{\tau<\infty} u(f_\tau)\, d\sigma + \varepsilon M. \square$$

Proposition 2 follows from Lemmas 2 and 3 and Theorem 2 follows from Propositions 1 and 2.

Despite Theorem 2, it can happen that there is an optimal stationary plan for the lim sup problem and not for the lim inf problem (cf. [4, Example 7.3] or the example of section 2).

For a leavable problem in which u is nonnegative and unbounded, there may be an optimal stationary plan for the lim sup payoff and yet the collection of stationary plans may not be uniformly adequate for the lim inf payoff. Here is an example based on one of Blackwell [1].

Example. Let $F = \{0,1,2,\ldots\}$; $u(0) = 0$, $u(2n) = 2^n - n$ for $n = 1,2,\ldots$, $u(2n+1) = 0$ for $n = 0,1,\ldots$; $\Gamma(0) = \{\delta(0)\}$, $\Gamma(2n) = \{\delta(2n),\delta(2n+1)\}$ for $n = 1,2,\ldots$; $\Gamma(2n+1) = \{\delta(2n+1),\frac{1}{2}(\delta(2n+2)+\delta(0)\}$. Then $V = W$ and $V(0) = 0$, $V(2n) = V(2n+1) = 2^n$ for $n = 1,2,\ldots$. An optimal plan γ^∞ for the lim sup payoff uses the selector γ where $\gamma(n) = \delta(n+1)$ for all $n > 0$. However, this plan has utility 0 for the lim inf payoff. In fact, it is easy to see that any ε-optimal stationary plan β^∞ must satisfy $\beta(n) = \delta(n+1)$ for large n and thus would have lim inf utility 0 for sufficiently large n. So there is no such plan.

This example may be misleading since the appropriate notion of

ε-optimality for nonnegative u is the multiplicative one [5].

5. Other payoff functions.

Let g be a bounded, real-valued function defined on H which is measurable either in the conventional sense as in [4] and [12] or in the snese of [7]. Define the optimal return function W_g for the generalized gambling problem (F,Γ,g) to be

$$W_g(f) = \sup\{\int g\, d\sigma : \sigma \in \Sigma(f)\}.$$

If g is shift-invariant, then results similar to many of those in section 2 can be proved ([4] and [12]). Even for a general g, some results are available [13].

A particularly interesting theorem due to Hill [6] is that, if F is finite and g is both shift-invariant and permutation-invariant, then ε-optimal Markov strategies are available. It is natural to ask for such g whether optimal stationary families are available for finite problems (i.e., when F is finite and $\Gamma(f)$ is finite for all f). A trivial example shows this is not the case even for leavable, deterministic Γ. (The example was discovered during a conversation with Hans Bodewig.)

Example. Let $F = \{0,1,2\}$; $\Gamma(0) = \{\delta(0),\delta(1),\delta(2)\}$, $\Gamma(1) = \{\delta(0),\delta(1)\}$, $\Gamma(2) = \{\delta(0),\delta(2)\}$; $g = u_1^* + u_2^*$ where u_i is the indicator of $\{i\}$ for $i = 1,2$. It is trivial to construct a $\sigma \in \Sigma(0)$ such that $\int g\, d\sigma = 2$, but $\int g\, d\sigma \leq 1$ for every stationary σ. If g is defined instead to be the minimum of u_1^* and u_2^*, then again stationary plans are inadequate.

It would be interesting to characterize the class of those g's for which optimal stationary plans always exist for finite problems. The example shows that this class of functions is neither a linear space nor a lattice.

If g is shift-invariant, the problem of the existence of optimal stationary plans for finite problems can be reduced to that of the existence of equalizing stationary plans just as was done above for the special case $g = u_*$.

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